

# Semiclassical approach for multiparticle production in scalar theories

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## Abstract

We propose a semiclassical approach to calculate multiparticle cross sections in scalar theories, which have been strongly argued to have the exponential form  $\exp(\lambda^{-1}F(\lambda n, \epsilon))$  in the regime  $\lambda \rightarrow 0$ ,  $\lambda n, \epsilon = \text{fixed}$ , where  $\lambda$  is the scalar coupling,  $n$  is the number of produced particles, and  $\epsilon$  is the kinetic energy per final particle. The formalism is based on singular solutions to the field equation, which satisfy certain boundary and extremizing conditions. At low multiplicities and small kinetic energies per final particle we reproduce in the framework of this formalism the main perturbative results. We also obtain a lower bound on the tree-level cross section in the ultra-relativistic regime.

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# 1 Introduction

Recently, the problem of multiparticle production in weakly coupled scalar field theories has received close attention. This problem has been initiated by the qualitative observation [1, 2] that in the  $\frac{\lambda}{4}\phi^4$  theory at tree level the probability of processes producing large number of bosons exhibit factorial dependence on the multiplicity of the final state. This dependence originates from the large number of tree graphs contributing to the process of multiparticle production: at multiplicity  $n \gg 1$  the number of graphs is of order  $n!$ . At  $n \sim 1/\lambda$  this factor is sufficient to compensate the suppression due to the smallness of the coupling constant, and the tree-level multiparticle cross sections become large. Much efforts have been made to understand how this behavior is changed by loop corrections, though a conclusive result on this issue is still lacking right now.

So far, quantitative calculations have been performed mostly at, or near, the multiparticle threshold, where some perturbative techniques have been developed and extensively explored. The tree amplitude of transition from one initial virtual particle to  $n$  real bosons at rest (the  $1 \rightarrow n$  process) can be computed either by summing Feynman diagrams or by using some appropriate classical solution [3, 4, 5] and the result reads

$$A_n^{tree}(0) = n! \left( \frac{\lambda}{8} \right)^{\frac{n-1}{2}} \quad (1)$$

(the boson mass in this formula and further is set to 1). The same methods have been applied for calculating the amplitude beyond the threshold or beyond the tree level [6]. In the first case it has been found that when the final particles are non-relativistic, the tree amplitude is an exponent of the total kinetic energy of final particles,

$$A_n^{tree}(\epsilon) = A_n^{tree}(0) e^{-\frac{5}{6}n\epsilon} \quad (2)$$

where  $A_n^{tree}(0)$  is given by eq.(1) and  $\epsilon$  is the kinetic energy per particle in the final state. The exponential fall of the tree amplitude beyond threshold in eq.(2) is not sufficient, however, to make the cross section small. The second result concerns loop corrections at exact threshold and reads that the leading- $n$  contributions from each loop level (namely, the  $\lambda n^2$  contribution from the first loop,  $\lambda^2 n^4$  from the second and  $\lambda^k n^{2k}$  from the  $k$ -th) sum up to an exponent, so at not very large  $n$ , (when subleading on  $n$  contributions can be neglected, presumably at  $\lambda n \ll 1$ ) the  $1 \rightarrow n$  amplitude at threshold has the form,

$$A_n(0) = A_n^{tree}(0) e^{B\lambda n^2} \quad (3)$$

where  $B$  is a constant that depends on the number of spatial dimensions,

$$B = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{9}{8\omega_{\mathbf{k}}(\omega_{\mathbf{k}}^2 - 1)(\omega_{\mathbf{k}}^2 - 4)}, \quad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + 1}$$

In particular, in (3+1) dimensions ( $d = 3$ ), the numeric value of  $B$  is

$$B = -\frac{1}{64\pi^2}(\ln(7 + 4\sqrt{3}) - i\pi)$$

The physically interesting quantity is however not the amplitude, but the cross section, or transition rate. For the  $1 \rightarrow n$  process near threshold this quantity is easy to evaluate at  $n \ll 1/\lambda$ , having on hand the two results above. In fact, if  $\epsilon$  is small enough for the amplitude to be constant in the whole phase volume, the cross section is equal to

$$\sigma(E, n) = |A_n(\epsilon)|^2 V_n$$

where  $A_n(\epsilon) = A_n^{tree} \exp(-\frac{5}{6}n\epsilon + B\lambda n^2)$ , and  $V_n$  is the bosonic phase volume. An important point to note is that at large  $n$ ,  $V_n$  has the exponential form except from the factor of  $1/n!$ ,

$$V_n \propto \frac{1}{n!} \exp\left(\frac{dn}{2} \left(\ln \frac{\epsilon}{\pi d} + 1\right) + \frac{d-2}{4}n\epsilon + O(n\epsilon^2)\right) \quad (4)$$

and now it is easy to verify that the cross section is exponential,

$$\sigma(E, n) \propto \exp\left(n \ln \frac{\lambda n}{16} - n + \frac{dn}{2} \left(\ln \frac{\epsilon}{\pi d} + 1\right) + \left(\frac{d-2}{4} - \frac{5}{3}\right)n\epsilon + 2\text{Re}B\lambda^2 n\right) \quad (5)$$

Though eq.(5) is valid only at small  $\epsilon$  and  $\lambda n$ , the form of  $\sigma(E, n)$  strongly supports the hypothesis that in the most interesting regime  $\lambda n \sim 1$ ,  $\epsilon \sim 1$  the cross section is also exponential [6].

$$\sigma(E, n) \propto \exp\left(\frac{1}{\lambda}F(\lambda n, \epsilon)\right), \quad \epsilon = \frac{E - n}{n} \quad (6)$$

Moreover, there are indications [7] that the exponent  $F(\lambda n, \epsilon)$  is independent of the few-particle initial state (i.e. the cross section of  $2 \rightarrow n$ ,  $3 \rightarrow n$ , etc. processes coincide, with exponential accuracy, with that of  $1 \rightarrow n$ ). The function  $F(\lambda n, \epsilon)$  is unknown, but some terms of its expansion at small  $\lambda n$  and  $\epsilon$  can be found from (5),

$$\begin{aligned} F(\lambda n, \epsilon) = & \lambda n \ln \frac{\lambda n}{16} - \lambda n + \frac{d}{2} \left(\ln \frac{\epsilon}{\pi d} + 1\right) \lambda n + \left(\frac{d-2}{4} - \frac{5}{3}\right) \lambda n \epsilon + 2\text{Re}B\lambda^2 n^2 + \\ & + O(\lambda^3 n^3) + O(\lambda^2 n^2 \epsilon) + O(\lambda n \epsilon^2) \end{aligned} \quad (7)$$

The situation that has emerged here shows a complete analogy to the instanton-like processes at high energies. As in the latter case, the exponential form of the multiparticle cross sections is a strong argument in favor to the semiclassical calculability of the exponent  $F$ , but says nothing about the nature of possible calculation schemes.

In this paper we propose a semiclassical method to calculate the multiparticle cross section. By using the coherent state formalism, we reduce the calculation to the problem of solving classical field equation with certain boundary conditions in the asymptotic regions  $t \rightarrow \pm\infty$ . The technique, as well as the boundary value problem are very similar to the those in the case of instanton transitions. The most essential difference from the latter case is that the boundary value problem for multiparticle production possesses only singular solutions. In particular, the field configuration defining the cross section is the one that is singular at one point  $t = \mathbf{x} = 0$  and regular elsewhere in the Minkowskian space-time. Note that some other approaches utilizing singular solutions have been proposed recently for both multiparticle processes in scalar theories and instanton-like transitions [8, 9, 10, 11], most being inspired by the Landau procedure for calculating the semiclassical matrix elements [12]. We emphasize, however, that in the framework of our formalism the singular field configuration is determined in a unique way by the boundary conditions and the structure of its singularity in the Minkowskian space-time.

The field configuration with the required properties can also be found in a different setting. Namely, if one makes analytical continuation to the complex times and looks for solutions to the boundary value problem which is singular on some surface (in the simplest case the surface lies in the Euclidean space-time) and extremizes the transition rate over all possible forms of this surface, one obtains the same field configuration as one would find by solving the boundary value problem. Actually, this formalism sometimes appears to be simpler and will be applied for making quantitative calculations in this paper, which include reproducing eq.(5) at small  $\lambda n$  and  $\epsilon$  and finding a lower bound on the tree-level cross section in the ultra-relativistic limit of the final state.

This paper is organized as follows. In Sect.2 we derive the classical problem for the multiparticle cross sections. Sect.3 is devoted to the tree amplitude at threshold. A simplified, entirely Euclidean version of the classical problem is presented and the perturbative result for the energy dependence of the tree amplitude near threshold is reproduced. In the opposite, ultra-relativistic, limit, we obtain a lower bound on the tree cross section. In Sect. 4 we consider the amplitude (with loops) at exact threshold, where the procedure for calculating the amplitude is derived from the general formalism. We reproduce the exponentiation factor coming from leading- $n$  loop in the limit  $\lambda n \ll 1$ .

Finally, Sect. 5 contains concluding remarks.

## 2 General formalism

### 2.1 The boundary value problem

We consider the scalar field theory without symmetry breaking in  $(d + 1)$ -dimensional space-time,

$$S = \int d^{d+1}x \left( \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \right)$$

The quantity in interest is the total transition rate from an initial few-particle state to all possible final states having given energy  $E$  and multiplicity  $n$ . One writes,

$$\sigma(E, n) = \sum_f |\langle f | P_E P_n \hat{S} \hat{A} | 0 \rangle|^2 \quad (8)$$

where  $\hat{A}$  is the operator that creates the initial state from the vacuum,  $\hat{S}$  is the  $S$ -matrix,  $P_E$  and  $P_n$  are the projection operators to states with energy  $E$  and number of particles  $n$ , and the sum runs over all final states  $|f\rangle$ . Different choices of the operator  $\hat{A}$  corresponds to different initial states: for example,  $\hat{A} = \phi$  corresponds to the  $1 \rightarrow n$  process (for the operator  $\hat{A}$  describing the  $2 \rightarrow n$  process see [7]). We recall perturbative calculations in [7] indicating that  $F$  does not depend on the particular choice of  $\hat{A}$ , providing the latter is independent of  $\lambda$  parametrically. Making use of this fact, we will calculate (8) for the operator  $\hat{A}$  most convenient for our purpose. Namely, we choose  $\hat{A}$  in the following exponential form,

$$\hat{A} = e^{j\phi(0)}$$

where  $j$  is some arbitrary number.

Following the technique of [13], we derive the classical boundary value problem for  $\sigma(E, n)$ . Using the coherent state formalism [14], one rewrites eq.(8) in the following integral form,

$$\begin{aligned} \sigma(E, n) = & \int db_{\mathbf{k}}^* db_{\mathbf{k}} d\xi d\eta \mathcal{D}\phi \mathcal{D}\phi' \exp \left( - \int d\mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\xi + i\eta} + iE\xi + in\eta + \right. \\ & \left. + B_i(0, \phi_i) + B_f(b_{\mathbf{k}}^*, \phi_f) + B_i^*(0, \phi_i') + B_f^*(b_{\mathbf{k}}, \phi_f') + iS[\phi] - iS[\phi'] + j\phi(0) + j\phi'(0) \right) \quad (9) \end{aligned}$$

In eq.(9),  $B$ 's stay for the boundary terms,

$$B_i(0, \phi_i) = -\frac{1}{2} \int d\mathbf{k} \omega_{\mathbf{k}} \phi_i(\mathbf{k}) \phi_i(-\mathbf{k})$$

$$B_f(b_{\mathbf{k}}^*, \phi_f) = -\frac{1}{2} \int d\mathbf{k} b_{\mathbf{k}}^* b_{-\mathbf{k}}^* e^{2i\omega_k T_f} + \int d\mathbf{k} \sqrt{2\omega_k} b_{\mathbf{k}}^* \phi_f(\mathbf{k}) e^{i\omega_k T_f} - \frac{1}{2} \int d\mathbf{k} \omega_k \phi_f(\mathbf{k}) \phi_f(-\mathbf{k})$$

where  $T_f$  is some final time moment (the limit  $T_f \rightarrow +\infty$  is assumed), while  $\phi_i(\mathbf{k})$  and  $\phi_f(\mathbf{k})$  are the Fourier transformations of the field in the initial and final asymptotic regions.

It is easy to see that if one takes  $j$  to be of order  $1/\sqrt{\lambda}$ , then the integral (9) has the semiclassical nature in the limit  $\lambda \rightarrow 0$ ,  $\lambda E$ ,  $\lambda n = \text{fixed}$ , and one can expect that it is saturated by a saddle point, where  $\phi$ ,  $\phi'$ ,  $b_{\mathbf{k}}$  and  $b_{\mathbf{k}}^*$  are of order  $1/\sqrt{\lambda}$ , and  $\xi$  and  $\eta$  are of order 1. In this case, the cross section certainly has the exponential form (6). However, the assumption  $j \sim 1/\sqrt{\lambda}$  contradicts the requirement that  $j$  does not depend on  $\lambda$ , which forces us to take  $j$  parametrically smaller,  $j \sim 1$ . This difficulty is obviously the consequence of the non-semiclassical nature of the initial state that contains few energetic particles, in contrast with the final state consisting of many soft ones. A method to overcome this difficulty has been suggested for the instanton transitions [15, 16] and can be applied for our problem in an analogous way. In this method one evaluates the integral (9) for  $j = \alpha/\sqrt{\lambda}$ , where  $\alpha$  is some constant, in saddle-point approximation, and find the cross section in the exponential form  $\sigma \sim e^W$ , where  $W$  is of order  $1/\lambda$  and depends on  $\alpha$ . After that one takes the limit  $\alpha \rightarrow 0$ . The claim is that in this limit one reproduces the value of  $W$  at  $j \sim 1$ .

Apparently, the weak point in this way of reasoning is the limit  $\alpha \rightarrow 0$ : it is neither obvious that this limit is smooth, nor that it reproduces the amplitude with one initial particle. Moreover, the exponent of the cross section of the  $1 \rightarrow n$  process contains contribution from *loops* (the term  $2\text{Re}B\lambda n^2$  in eq.(7)) that seems to be of quantum, rather than classical, nature. Nevertheless, it turns out that this nontrivial exponentiated loops are reproduced by the semiclassical calculations (see Sect.4 below), which is a sound argument in favor to the hypothesis about the  $\alpha \rightarrow 0$  limit.

Therefore, we assume that  $j \sim 1/\sqrt{\lambda}$  and look for the saddle point of the exponent of the integrand in (9). The saddle-point equations can be divided into two groups. The first group contains the equations for  $\phi$ ,

$$\frac{\delta S}{\delta \phi} = ij\delta(x) \quad (10)$$

$$i\dot{\phi}_i(\mathbf{k}) + \omega_k \phi_i(\mathbf{k}) = 0 \quad (11)$$

$$i\dot{\phi}_f(\mathbf{k}) - \omega_k \phi_f(\mathbf{k}) + \sqrt{2\omega_k} b_{-\mathbf{k}}^* e^{i\omega_k T_f} = 0 \quad (12)$$

$$-b_{\mathbf{k}} e^{i\omega_k \xi + i\eta} - b_{-\mathbf{k}}^* e^{2i\omega_k T_f} + \sqrt{2\omega_k} \phi_f(\mathbf{k}) e^{i\omega_k T_f} = 0 \quad (13)$$

and similar equations for  $\phi'$ . The equations from the second group relate the energy  $E$  and the number of final particles  $n$  to other parameters,

$$E = \int d\mathbf{k} \omega_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\xi + i\eta} \quad (14)$$

$$n = \int d\mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\xi + i\eta} \quad (15)$$

First, let us consider the equations for  $\phi$ . Eq.(10) is simply the field equation with a  $\delta$ -like source, while eqs.(11), (12), (13) can be considered as boundary conditions in initial and final asymptotics  $t \rightarrow \mp\infty$ . It is convenient to rewrite these boundary conditions in a more transparent form. For this end we note that since  $\phi$  is a superposition of plane waves in the limit  $t \rightarrow -\infty$  and  $t \rightarrow +\infty$ , eq.(11) can be satisfied only if the initial asymptotics of  $\phi$  is purely Feynman,

$$\phi_i(\mathbf{k}) = a_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}}t}, \quad t \rightarrow -\infty \quad (16)$$

where  $a_{\mathbf{k}}$  are arbitrary Fourier components, while eq.(13) implies the following asymptotics of  $\phi$  in the final asymptotic region,

$$\phi_f(\mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}\xi + i\eta - i\omega_{\mathbf{k}}t} + b_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}}t} \right), \quad t \rightarrow +\infty \quad (17)$$

It is easy to see that eq.(17) satisfies the condition (12) automatically.

Let us turn to the equations from the second group. The physical meaning of eqs.(14) and (15) is simple: they read that  $E$  and  $n$  are the energy and the number of particles of the field  $\phi$  in its final asymptotics (17). Since  $\phi$  satisfies the sourceless field equation at all values of  $t$  but  $t = 0$  (where the source is not vanishing), the energy of the field conserves in the two regions  $t > 0$  and  $t < 0$  separately, but may have discontinuity at  $t = 0$ . The energy in the region  $t > 0$  is equal to that in the limit  $t \rightarrow +\infty$  and therefore is  $E$ . To find the energy at  $t < 0$  we make use of the  $\phi$ 's initial asymptotics, and since the latter contains only Feynman components, it vanishes. So, we find that the energy has a finite jump at  $t = 0$  which, naturally, is associated with the  $\delta$ -functional source located at  $t = \mathbf{x} = 0$ .

To simplify further discussions, let us make two conjectures that, as we will see in what follows, do not lead to contradiction. The first is that the saddle-point values of  $b_{\mathbf{k}}$  and  $b_{\mathbf{k}}^*$  are complex conjugated to each other. The physical meaning of this assumption is that the sum over final states in eq.(8) is saturated by a single coherent state. The second conjecture is that the saddle point values of  $\xi$  and  $\eta$  are purely imaginary,

$$\xi = -iT, \quad \eta = i\theta$$

where  $T$  and  $\theta$  are real (for further convenience we choose different sign conventions for  $T$  and  $\theta$ ).

With the two conjectures formulated above, the field configuration describing the multiparticle process at given energy  $E$  and multiplicity  $n$  is the solution to the field equation with source

$$\frac{\delta S}{\delta \phi} = ij\delta^{d+1}(x) \quad (18)$$

with the two boundary conditions,

$$\phi(\mathbf{k}) = a_{\mathbf{k}}e^{i\omega_k t}, \quad t \rightarrow -\infty \quad (19)$$

$$\phi(\mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( b_{\mathbf{k}}e^{\omega_k T - \theta - i\omega_k t} + b_{-\mathbf{k}}^*e^{i\omega_k t} \right), \quad t \rightarrow +\infty \quad (20)$$

It is easy to notice that the initial boundary conditions (19) can be reformulated in Euclidean language. In fact, making the Wick rotation to the Euclidean time  $\tau = -it$ , eq.(19) reads that  $\phi(\mathbf{k})$ , as a function of  $\tau$ , contains only the decaying component in the asymptotics  $\tau \rightarrow +\infty$ ,

$$\phi(\mathbf{k}) = a_{\mathbf{k}}e^{-\omega_k \tau}, \quad \tau \rightarrow +\infty \quad (21)$$

In contrast, the the final asymptotics (20) contains both frequencies and cannot be rewritten in Euclidean language. Moreover the field in the final asymptotics is, in general, complex.

The boundary value problem for  $\phi'$  can be derived in analogous way,

$$\frac{\delta S}{\delta \phi} = -ij\delta^{d+1}(x)$$

$$\phi'(\mathbf{k}) = a_{\mathbf{k}}e^{-i\omega_k t}, \quad t \rightarrow -\infty$$

$$\phi'(\mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( b_{\mathbf{k}}e^{-i\omega_k t} + b_{-\mathbf{k}}^*e^{\omega_k T - \theta + i\omega_k t} \right), \quad t \rightarrow +\infty \quad (22)$$

Notice that if  $\phi$  is a solution to the boundary value problem (18, 19 20), its complex conjugate  $\phi^*$  satisfies eqs.(22). This fact simplifies our calculations, since we need to solve only one boundary value problem instead of two. In further discussions we will assume that  $\phi' = \phi^*$ .

The saddle point of the integral (9) determines the cross section, which has the exponential form,

$$\sigma(E, n) \sim e^{W(E, n)}$$

where

$$W(E, n) = \frac{1}{\lambda} F(\lambda n, \epsilon) = ET - n\theta - 2\text{Im}S[\phi] \quad (23)$$



The relation between  $E$ ,  $n$  and  $T$ ,  $\theta$ , can be found either from eqs.(14) and (15), or, equivalently, from

$$2\frac{\partial \text{Im}S}{\partial T} = E, \quad 2\frac{\partial \text{Im}S}{\partial \theta} = n. \quad (24)$$

which can be easily understood if one recalls that  $\xi = -iT$  and  $\eta = i\theta$  are the saddle point of the integrand in eq.(9). From (23) one sees that  $W(E, n)$  is the Legendre transformation of  $2\text{Im}S(T, \theta)$ , and therefore one obtains the following important relations,

$$\frac{\partial W}{\partial E} = T, \quad \frac{\partial W}{\partial n} = -\theta \quad (25)$$

Having derived the boundary value problem for calculating the transition rate at finite  $j$ , let us discuss the limit  $j \rightarrow 0$ . It can be shown that in this limit the field configuration becomes singular at  $x = 0$ . In fact, according to eq.(18),  $\phi$  has discontinuity at  $t = 0$ ,

$$\delta\dot{\phi}(\mathbf{x}) = \dot{\phi}(\mathbf{x})|_{t=+0} - \dot{\phi}(\mathbf{x})|_{t=-0} = ij\delta^d(\mathbf{x})$$

This discontinuity leads a jump of the energy at  $t = 0$ , since the latter contains the term  $\frac{1}{2} \int d\mathbf{x} \dot{\phi}^2$ . On the other hand, the discontinuity of the energy  $E$  is supposed to be finite while that of  $\dot{\phi}$  is proportional to  $j$  and tends to 0. When  $j$  is small one has

$$E = \int d\mathbf{x} \dot{\phi}(0, \mathbf{x}) \delta\dot{\phi}(0, \mathbf{x}) = ij\dot{\phi}(0)$$

One sees that when  $E$  is fixed and  $j \rightarrow 0$ ,  $\dot{\phi}(0)$  goes to infinity, which means that the field configuration becomes singular at  $x = 0$  in the limit of vanishing source. This is not surprising, since in the limit  $j \rightarrow 0$  eq.(10) becomes the sourceless field equation, whose regular solutions conserve the energy and therefore do not obey the boundary conditions.

So, to evaluate the transition rate one should find the solution to the field equation which obeys the boundary conditions and has singularity at  $t = \mathbf{x} = 0$ , but remains regular elsewhere in the Minkowskian space-time. For doing calculations, however, we will use another formulation of the boundary value problem.

## 2.2 Extremization procedure

Let us discuss the structure of singularities of our solution. Recall that in Minkowskian space-time,  $\phi$  is regular everywhere except a point-like singularity at  $x = 0$ . However, if one extrapolates  $\phi$  into Euclidean times, it may occur that  $\phi$  develops more singularities beside that at  $x = 0$ . Let us consider a simple possibility that  $\phi$  is singular on some  $d$ -dimensional surface  $A$  in the Euclidean space-time, which we will parametrize either

as  $x_\mu = x_\mu(s_i)$ , where  $s_i$  are  $d$  coordinates on the surface, or  $\tau = \tau_0(\mathbf{x})$ . In the region near  $A$ ,  $\phi$  is inverse to the distance to the surface,

$$\phi \sim \sqrt{\frac{2}{\lambda}} \frac{1}{l(x)} \quad (26)$$

where  $l(x)$  is the distance from the point  $x$  to  $A$ . In fact eq.(26) is the only possible form of the leading singularity of  $\phi$  in the region near the singularity surface  $A$ . In fact, one can even develop a perturbation theory on  $l$  and find the correction to eq.(26), which is of order  $l(x)$ . However, one will see soon that ambiguity begins at the order of  $O(l^3)$  (on other words, the terms higher than  $O(l^3)$  is not defined uniquely by the form of  $A$ ), which reflect the fact that the solution is not defined uniquely by the surface where it is singular. If two solutions are singular on the same surface  $A$ , the difference between them goes to 0 when one approaches the surface as  $l^3$ . Note that if  $A$  touches the plane  $\tau = 0$  only at one point  $x = 0$ , then in Minkowskian space-time  $\phi$  has the required structure, i.e. is singular only at  $x = 0$ .

Let us take an arbitrary surface  $A$  satisfying the latter requirement and determine a field configuration  $\phi$ , which consists of two parts  $\phi_1$  and  $\phi_2$ , as follows. Both  $\phi_1$  and  $\phi_2$  are supposed to be solutions to the field equation and singular on  $A$ , but each of them obeys one boundary condition from eqs.(19) and (20). Namely, we require that  $\phi_1$  decreases in the Euclidean asymptotics,

$$\phi_1 \rightarrow 0, \quad \tau \rightarrow +\infty$$

while  $\phi_2$  obeys the boundary condition in the Minkowskian limit,

$$\phi_2(\mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( b_{\mathbf{k}} e^{\omega_{\mathbf{k}} T - \theta - i\omega_{\mathbf{k}} t} + b_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}} t} \right), \quad t \rightarrow +\infty$$

One may imagine the field  $\phi$  is defined on a particular contour on the complex time plane, which at each value of  $\mathbf{x}$  goes along the Euclidean time axis from  $i\infty$  to some  $i\tau_0(\mathbf{x})$  lying on the singularity surface and then goes back to 0 and then along the Minkowskian time axis to  $\infty$  (fig.1). The field  $\phi_1$  is defined on the first part of the contour,  $(i\infty, i\tau_0)$ , while  $\phi_2$  on the two final parts,  $(i\tau_0, 0)$  and  $(0, \infty)$ . Note that there is a region on the Euclidean time axis where both  $\phi_1$  and  $\phi_2$  are defined, namely  $(i\tau_0, 0)$ .

Despite the fact that  $\phi$  obeys the boundary conditions of the boundary value problem, it is not the solution to the latter, since  $\phi_1$  and  $\phi_2$  need not to be equal at  $t = 0$ . In other words, for a generic surface  $A$ ,  $\phi$  contains discontinuities on the whole plane  $t = 0$ , instead of being singular at one point  $t = \mathbf{x} = 0$ . On the other hand, if one manages to

choose the surface  $A$  in such a way that  $\phi_1(0, \mathbf{x}) = \phi_2(0, \mathbf{x})$  for any  $\mathbf{x} \neq 0$ ,  $\phi$  would be the solution to the boundary value problem.

Let us define the Euclidean action of  $\phi$  as the sum of the action of  $\phi_1$  and  $\phi_2$ ,  $S[\phi_1]$  and  $S[\phi_2]$ , each calculated along the corresponding part of the contour,

$$S_E[\phi] = - \int_{+\infty}^{\tau_0(\mathbf{x})} d\tau d\mathbf{x} \left( \frac{1}{2} (\partial_\mu \phi_1)^2 + V(\phi_1) \right) - \int_{\tau_0(\mathbf{x})}^0 d\tau d\mathbf{x} \left( \frac{1}{2} (\partial_\mu \phi_2)^2 + V(\phi_2) \right) - i \int_0^\infty dt d\mathbf{x} L(\phi_2)$$

Since  $\phi_{1,2}$  are singular at  $\tau = \tau_0(\mathbf{x})$ , both  $S[\phi_1]$  and  $S[\phi_2]$  are infinite, but since the integration contour for  $\phi_1$  and  $\phi_2$  goes at different directions near the singularities, one can hope that their sum  $S[\phi]$  is nevertheless finite.

Now we will show that the “correct” singularity surface  $A$  determined by the condition that  $\phi_1 = \phi_2$  at  $t = 0$  can be found by extremizing the real part of the Euclidean action  $S_E[\phi]$  with respect to all possible form of the surface  $A$ , with the requirement that the point  $\tau = \mathbf{x} = 0$  lies on the latter. First let us regularize  $\phi$  to avoid dealing with infinities in intermediate calculations. For this end we replace the condition that  $\phi$  is singular on  $A$  by the condition that  $\phi = \phi_0$  on the same surface, where  $\phi_0$  is some large, but finite number, which will eventually tends to infinity. So, we set  $\phi_1 = \phi_2 = \phi_0$  on  $A$ . Since  $\phi_1$  and  $\phi_2$  are, in general, different in the region near the surface, we expect that the derivatives of  $\phi_{1,2}$  are different on  $A$ . Let us denote

$$\partial_n(\phi_1 - \phi_2) = j(s_i) \quad (27)$$

where  $\partial_n$  is the derivative along the direction normal to  $A$ . The configuration  $\phi$ , thus, can be regarded as the solution to the field equation with a source that is distributed over the surface  $A$ ,

$$\frac{\partial S_E}{\partial \phi} = j(x) = \int ds_i j(s_i) \delta(x - x_\mu(s))$$

(in this formula we assume that the appropriate metric factor has been included in  $ds_i$ ). Now, as an intermediate step, we show that once  $S_E[\phi]$  has been extremized with respect to  $A$ , the source  $j(x)$  is proportional to  $\delta(x)$  (in other words, the source is located at the point  $x = 0$  rather than distributed over  $A$ ). Let us take an arbitrary singularity surface  $A$  and deform it slightly to  $A'$ . This can be represented as shifting each point  $x_\mu$  on  $A$  by a small vector  $\delta x_\mu = n_\mu \delta x$ , where  $n_\mu$  is the unit vector perpendicular to  $A$ , so that  $x_\mu + \delta n_\mu$  lies on  $A'$ . To ensure that  $x = 0$  is always a singular point, we require that  $\delta x_\mu|_{x=0} = 0$  (fig.2).

The new surface  $A'$  thus corresponds to new configurations  $\phi'_{1,2} = \phi_{1,2} + \delta \phi_{1,2}$ . We will evaluate the variation of  $S[\phi_1]$  and  $S[\phi_2]$  separately. The variation of  $S[\phi_1]$  is due to

two factors: the first is the variation of the field,  $\phi_1 \rightarrow \phi_1 + \delta\phi_1$  and the second is the change of the integration region. The first contribution can be reduced to a boundary integral, since  $\phi_1$  is a solution to the field equation,

$$-\int_{\infty}^{\tau_0(\mathbf{x})} d\tau d\mathbf{x} \delta \left( \frac{1}{2} (\partial_\mu \phi_1)^2 + V(\phi_1) \right) = -\int_A dx [\partial_\mu \phi_1 \partial_\mu \delta\phi_1 - V'(\phi_1) \delta\phi_1] = -\int_A ds (\partial_\mu \phi_1 \cdot n_\mu) \delta\phi_1$$

If  $\delta x_\mu$  is small, the second contribution that is associated with the change of the integration region can be also reduced to a boundary integral, where each point on the surface  $A$  is integrated with the weight  $\delta x$ . Therefore, the full variation of  $S[\phi_1]$  is

$$\delta S_E[\phi_1] = \int_A ds \left[ (\partial_\mu \phi \cdot n_\mu) \delta\phi - \left( \frac{1}{2} (\partial_\mu \phi_1)^2 + V(\phi_1) \right) \delta x(s) \right]$$

Let us make use of the fact that  $\phi_1 = \phi_0$  on  $A$ , and  $\phi'_1 = \phi_0$  on  $A'$ . We have

$$\delta\phi(x_\mu) = \phi'(x_\mu) - \phi(x_\mu) = \phi'(x_\mu) - \phi'(x_\mu + \delta x_\mu) = -(\partial_\mu \phi \cdot n_\mu) \delta x(s)$$

Therefore,

$$\begin{aligned} \delta S[\phi_1] &= \int_A ds \left[ (\partial_\mu \phi_1)^2 - \left( \frac{1}{2} (\partial_\mu \phi_1)^2 + V(\phi_1) \right) \right] \delta x(s) \\ &= \int_A ds \left[ \frac{1}{2} (\partial_n \phi_1)^2 - V(\phi_1) \right] \end{aligned} \quad (28)$$

where we have made use of the fact that on  $A$  the derivatives of  $\phi$  along directions tangent to  $A$  vanish (since  $\phi$  is constant on  $A$ ).

Let us turn to the variation of  $S[\phi_2]$ . The computation is completely analogous to the case of  $S[\phi_1]$ , with the exception that now there is a boundary term at  $t = +\infty$ . So we have,

$$\delta S_E[\phi_2] = -\int ds \left[ \frac{1}{2} (\partial_n \phi_2)^2 - V(\phi_2) \right] - i \int d\mathbf{x} \partial_0 \phi_2 \delta\phi_2|_{t=+\infty} \quad (29)$$

However one can show that the boundary term at  $t = +\infty$  is purely imaginary. In fact, since  $\phi_2$  and  $\phi'_2$  obey the boundary condition (20) with the same  $T$  and  $\theta$ , we have at  $t \rightarrow \infty$ ,

$$\delta\phi_2(\mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( \delta b_{\mathbf{k}} e^{\omega_{\mathbf{k}} T - \theta - i\omega_{\mathbf{k}} t} + \delta b_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}} t} \right), \quad t \rightarrow +\infty$$

Substituting this, as well as the asymptotics for  $\phi_2$ , to the boundary term at  $t = +\infty$ , we see that the latter term in the r.h.s. of eq.(29) is in fact purely imaginary

$$i \int d\mathbf{x} \partial_0 \phi_2 \delta\phi_2 = \int \frac{d\mathbf{k}}{(2\pi)^d} (b_{\mathbf{k}} \delta b_{\mathbf{k}}^* - b_{\mathbf{k}}^* \delta b_{\mathbf{k}}) e^{\omega_{\mathbf{k}} T - \theta}$$

and since we are interested only in the real part of the Euclidean action, this term can be dropped in further calculations. Taking the sum of (28) and (29), one finds,

$$\delta \text{Re} S_E[\phi] = \frac{1}{2} \int_A ds \left( (\partial_n \phi_1)^2 - (\partial_n \phi_2)^2 \right) \delta x(s)$$

Now when one takes  $\phi_0$  to be large, both  $\partial_n \phi_{1,2}$  are large but the difference between them is small. Making use of eq.(27) one writes

$$\delta \text{Re} S[\phi] = \int ds (\partial_n \phi) \delta x(s) j(s)$$

Now we see that the requirement that  $\delta \text{Re} S[\phi] = 0$  for all variations  $A$  obeying  $\delta x|_{x=0} = 0$  can be satisfied only if  $j(s)$  is proportional to the delta function,  $j(x) = j_0 \delta(x)$ . So, when we extremize the real part of the action  $S_E[\phi]$ , varying the surface  $A$ , the source  $j$ , which is at first distributed over  $A$ , gather to a localized delta-functional source  $j(x) = j_0 \delta(x)$

Suppose that we have performed this extremization procedure. Since now  $\phi_1$  and  $\phi_2$  are equal on  $A$  and their normal derivatives are also equal (except from the point  $x = 0$ ), these fields coincide in the region where they are both determined, namely, at  $\tau_0(\mathbf{x}) < \tau < 0$ . In particular,  $\phi_1 = \phi_2$  everywhere on the plane  $\tau = 0$  but  $\tau = \mathbf{x} = 0$ .

So far we have been dealing with the regularized field configurations. Let us now take the limit  $\phi_0 \rightarrow \infty$ . The coincidence of  $\phi_1$  and  $\phi_2$  at  $t = 0$  remains in this limit, however now the point  $t = \mathbf{x} = 0$  becomes singular. The strength of the delta-functional source,  $j_0$ , goes to 0 in order to keep the jump of energy finite. So,  $\phi$  obeys the sourceless field equation in Minkowskian space-time, and is singular only at  $x = 0$ , thus it is the solution to the boundary value problem.

To summarize, we have shown that the solution to the boundary value problem can be found by extremizing the real part of the Euclidean action  $S_E[\phi]$  (or the imaginary part of the Minkowskian action) over all singularity surfaces  $A$  containing the point  $t = \mathbf{x} = 0$ . Let us now apply this formalism to find the cross section in various limiting cases.

### 3 Tree-level cross sections

#### 3.1 General consideration

Consider the exponent for the cross section,  $F(\lambda n, \epsilon)$ , at small  $\lambda n$ . Keeping in mind the formula (7), one expects that at  $\epsilon \sim 1$ ,  $\lambda n \ll 1$ , the function  $F$  has the following form,

$$F(\lambda n, \epsilon) = \lambda n \ln \lambda n - \lambda n + \lambda n f(\epsilon) + O(\lambda^2 n^2) \quad (30)$$

where  $f(\epsilon)$  is some function of  $\epsilon$ . In what follows we will neglect the terms  $O(\lambda^2 n^2)$  and higher. Since these terms come from loops, it is equivalent to considering the tree level. Another way to see this is to compute the cross section corresponding to (30),

$$\sigma(E, n) \sim \exp\left(\frac{1}{\lambda} F(n\lambda, \epsilon)\right) \sim n! \lambda^n e^{nf(\epsilon)} \quad (31)$$

We see that the cross section depends on the coupling constant  $\lambda$  as  $\lambda^n$ , which is natural for tree diagrams whose number of vertices is  $n/2$ . Even at tree level, the cross section at arbitrary  $\epsilon$  has not been calculated (for lower bound see ref. [18]). In the non-relativistic limit  $\epsilon \ll 1$  the perturbative result (7) yields the following formula for  $f(\epsilon)$ ,

$$f(\epsilon) = -\ln 16 + \frac{d}{2} \ln \frac{\epsilon}{\pi d} + \frac{d}{2} + \left(\frac{d-2}{4} - \frac{5}{3}\right) \epsilon + O(\epsilon^2) \quad (32)$$

In this section we make no attempt to compute  $f(\epsilon)$  at arbitrary value of  $\epsilon$ . Our main goal is to reproduce eq.(32) from the formalism of Sect.2. We will also try to estimate the tree cross section from below in the limit  $\epsilon \rightarrow \infty$ .

Before considering the small- $\epsilon$  limit, let us first point out that the calculation procedure can be considerably simplified if one restricts himself to the tree level. We will work in the “extremization” formalism, not in the framework of the original Minkowskian boundary value problem, so we take an arbitrary surface  $A$  and calculate  $S_E[\phi]$  (we will deal only with the Euclidean action, so for simplicity further we will drop the index  $E$ ).

First note that from eqs.(25) and (30) one finds

$$\theta = -\frac{1}{\lambda} \frac{\partial F}{\partial n} = -\ln \frac{\lambda n}{16} - f(\epsilon)$$

So, in the limit  $\lambda n \rightarrow 0$ ,  $\theta \gg 1$  independent of the form of the function  $f(\epsilon)$ . We see that in the final asymptotics the Feynman part of  $\phi$ ,  $b_{\mathbf{k}} e^{-i\omega_k t + \omega_k T - \theta}$  is much smaller than the anti-Feynman part,  $b_{\mathbf{k}}^* e^{i\omega_k t}$ .

Recall that the initial asymptotics of  $\phi_1$  is the same as that of  $\phi$  eq.(21). By construction,  $\phi_1$  is defined on the first part of the contour of fig.1,  $(i\infty, 0)$ , however one can always analytically continue  $\phi_1$  to other parts. Let us investigate the difference between  $\phi_1$  and  $\phi_2$  on the parts 2 and 3. Denote

$$\tilde{\phi} = \phi_2 - \phi_1$$

so  $\tilde{\phi}$  is regular (in fact, equal to 0) on the surface  $A$  and obeys the boundary condition

$$\tilde{\phi}(\mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( b_{\mathbf{k}} e^{-i\omega_k t + \omega_k T - \theta} + (b_{-\mathbf{k}}^* - a_{-\mathbf{k}}^*) e^{i\omega_k t} \right)$$

in the final asymptotics. Let us show that the anti-Feynman part of  $\tilde{\phi}$ ,  $b_{-\mathbf{k}}^* - a_{-\mathbf{k}}^*$  is small and proportional to  $e^{-\theta}$ . Supposing that this is true, then the whole  $\tilde{\phi}$  is a small perturbation on  $\phi_1$  and thus obeys the linearized equation

$$(\partial_\mu^2 + 1 + 3\lambda\phi_1^2)\tilde{\phi} = 0 \quad (33)$$

with two boundary conditions

$$\tilde{\phi}|_A = 0$$

$$\tilde{\phi}(k) = \frac{1}{\sqrt{2\omega_k}} a_{\mathbf{k}} e^{-i\omega_k t + \omega_k T - \theta} + \text{any anti-Feynman part}, \quad t \rightarrow +\infty \quad (34)$$

(we have made use of the assumption that  $b_{\mathbf{k}} \approx a_{\mathbf{k}}$ ). Since the equation is linear and the final boundary conditions contains a factor of  $e^{-\theta}$ , the solution  $\tilde{\phi}$  is also proportional to  $e^{-\theta}$ , which is consistent with our starting assumption.

Now making use of the smallness of  $\tilde{\phi}$ , the action, up to the contributions of order  $e^{-\theta}$ , is equal to

$$S[\phi] = S_1[\phi_1] + S_{\{2,3\}}[\phi_1] - \int_{\{3\}} dx \left[ (\partial_\mu \phi_1)(\partial_\mu \tilde{\phi}) - V'(\phi_1)\tilde{\phi} \right] \quad (35)$$

It is easy to see that the action of  $\phi_1$  on the whole contour (the first two term in the r.h.s. of eq.(35)) vanishes (one can see this, for example, by making the Wick rotation of the third part of the contour). The last term in eq.(35) is reduces to boundary integrals. The boundary term on  $A$  is equal to 0 since near  $A$   $\tilde{\phi}(x)$  tends to 0 as  $l^3$ , where  $l$  is the distance from  $x$  to  $A$ , while  $\partial\phi_1 \sim l^{-2}$ . The boundary term at  $t = +\infty$  is

$$S[\phi] = i \int d\mathbf{x} \tilde{\phi} \partial_t \phi_1|_{t=+\infty} = \frac{1}{2} \int d\mathbf{k} a_{\mathbf{k}}^* a_{\mathbf{k}} e^{\omega_k T - \theta} \quad (36)$$

So, to compute the action, one need not really solve eq.(33): only the knowledge of  $\phi_1$  (more precisely, its Euclidean asymptotics) is required. Note that the Euclidean action is real.

Once the action is found, one should extremize (36) over all solutions that are singular at  $x = 0$ . In fact, one can see that one should extremize only  $\int d\mathbf{k} a_{\mathbf{k}}^* a_{\mathbf{k}} e^{\omega_k T}$ , since  $e^{-\theta}$  is just an overall factor in eq.(36). Moreover, since the action is bounded from below by 0, the extremum of the action is most likely the true minimum. We also expect that the action contains a classical factor  $1/\lambda$ , so, the extremized action has the following form

$$2S(T, \theta) = \frac{1}{\lambda} e^{g(T) - \theta} \quad (37)$$

where  $g(T)$  is some function. Eqs.(24) then read

$$E = 2 \frac{\partial S}{\partial T} = \frac{1}{\lambda} g'(T) e^{g(T) - \theta} \quad (38)$$

$$n = -2 \frac{\partial S}{\partial \theta} = \frac{1}{\lambda} e^{g(T) - \theta} \quad (39)$$

These equations should be solved with respect to  $T$  and  $\theta$ . Dividing (38) to (39), one obtains

$$1 + \epsilon = g'(T) \quad (40)$$

We see that the parameter  $T$  depends on  $\epsilon$  but not on  $\theta$ . Regarding (40) as an equation on  $T$ , we denote its solution as

$$T = T(\epsilon)$$

and from eq.(39) one finds  $\theta$  as a function of  $\epsilon$  and  $n$ ,

$$\theta = g(T(\epsilon)) - \ln(\lambda n)$$

Let us substitute the solution to the exponent of the cross section. One obtains

$$\frac{1}{\lambda} W = ET - n\theta - 2S(T, \theta) = n(1 + \epsilon)T(\epsilon) - n(g(T(\epsilon)) - \ln \lambda n) - n$$

Comparing the last equation with (30), one finds

$$f(\epsilon) = (1 + \epsilon)T(\epsilon) - g(T(\epsilon)) \quad (41)$$

Therefore, the problem of finding the tree cross section at any value of  $E$  and  $n$  can be formulated entirely in the Euclidean space-time. One looks for all solutions  $\phi_1(\tau, \mathbf{x})$  to the Euclidean field equations which are singular at  $\tau = \mathbf{x} = 0$  and decay as  $\tau \rightarrow +\infty$ , and calculates for each solution the corresponding Fourier components  $a_{\mathbf{k}}$  from its asymptotics at  $\tau \rightarrow \infty$ . Then one should maximize the integral  $\int d\mathbf{k} a_{\mathbf{k}}^* a_{\mathbf{k}} e^{\omega_{\mathbf{k}} T}$  and determines the function  $g(T)$  (eq.(37)). The required  $f(\epsilon)$  can be calculated using eq.(41), where  $T(\epsilon)$  is the solution to eq.(40)

Unfortunately, we are unable to carry out this program for arbitrary values of  $\epsilon$  due to the non-linearity of the field equation. At small  $\epsilon$  (which corresponds to non-relativistic final states) we will see that it reproduces the result found by perturbative calculations.



## 3.2 Non-relativistic regime

### 3.2.1 Leading order

At small  $\epsilon$ , the typical momentum of final particles is  $k_0 = \epsilon^{1/2}$  of much smaller than the mass, so one can expect that  $\phi_1$  is a slowly varying function of  $\mathbf{x}$ . The  $\mathbf{x}$ -independent solution to the field equation which is singular at  $\tau = 0$  and decays as  $\tau \rightarrow \infty$  is known explicitly,

$$\phi_1(\tau) = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh \tau}, \quad (42)$$

Let us restrict ourselves to the leading order. The field configuration in the first approximation can be obtained from (42) by a simple modification,

$$\phi_1(\tau, \mathbf{x}) = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh(\tau - \tau_0(\mathbf{x}))} \quad (43)$$

where  $\tau_0(\mathbf{x})$  is a slowly varying function of  $\mathbf{x}$ . One can check that eq.(43) satisfies the field equation to the accuracy of  $O((\partial_{\mathbf{x}}\tau_0)^2)$ . Note that  $\tau = \tau_0(\mathbf{x})$  is the surface of singularities of  $\phi_1$ . The point  $\tau = \mathbf{x} = 0$  should lie on the latter, so we require that  $\tau_0(0) = 0$ .

First, from eq.(43) one can relate  $a_{\mathbf{k}}$  to  $\tau_0(x)$ . At  $\tau \rightarrow \infty$ , the asymptotics of  $\phi_1$  in eq.(43) is

$$\phi(\tau, \mathbf{x}) = \sqrt{\frac{8}{\lambda}} e^{\tau_0(\mathbf{x})} e^{-\tau} \quad (44)$$

On the other hand,  $\phi_1$  can be expanded into plane wave in this region,

$$\phi_1 = \int \frac{d\mathbf{k}}{(2\pi)^{d/2} \sqrt{2\omega_k}} a_{-\mathbf{k}} e^{i\mathbf{k}\mathbf{x} - \omega_k \tau} \quad (45)$$

Recalling that typical momentum  $\mathbf{k}$  is small, as the first approximation one can replace  $\omega_{\mathbf{k}}$  by 1 and the r.h.s. of eq.(45) behaves like  $e^{-\tau}$ , as that of eq.(44). Comparing eqs.(44) and (45), one obtains,

$$\int \frac{d\mathbf{k}}{(2\pi)^{d/2}} a_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} = \sqrt{\frac{16}{\lambda}} e^{\tau_0(\mathbf{x})} \quad (46)$$

Taking in this equation  $\mathbf{x} = 0$ , and recalling the requirement  $\tau(\mathbf{x}) = 0$ , one finds the following constraint on  $a_{\mathbf{k}}$ ,

$$\int \frac{d\mathbf{k}}{(2\pi)^{d/2}} a_{\mathbf{k}} = \sqrt{\frac{16}{\lambda}} \quad (47)$$

To find  $W$  one should extremize the r.h.s of eq.(36) with the constraint (47). We follow the standard technique and introduce the term

$$C \left( \int \frac{d^d \mathbf{k}}{(2\pi)^{d/2}} a_{\mathbf{k}} - (2\pi)^{d/2} \sqrt{\frac{16}{\lambda}} \right)$$

where  $C$  is the Lagrange multiplier to the r.h.s. of eq.(36) and, varying with respect to  $a_{\mathbf{k}}^*$ , we obtain  $a_{\mathbf{k}}$ ,

$$a_{\mathbf{k}} = Ce^{-\omega_k T} \quad (48)$$

It is easy to notice that (48) in fact minimizes the action. The constant  $C$  can be determined from eq.(47),

$$C = \sqrt{\frac{16}{\lambda}} T^{d/2} e^T$$

Let us now calculate the function  $g(T)$ . According to eq.(37),

$$g(T) = \ln \lambda \left( \int d\mathbf{k} a_{\mathbf{k}}^* a_{\mathbf{k}} e^{\omega_k T} \right) = T + \frac{d}{2} \ln(2\pi T) + \ln 16$$

so eq.(40) becomes

$$1 + \frac{d}{2T} = 1 + \epsilon$$

whose solution is obviously

$$T(\epsilon) = \frac{d}{2\epsilon} \quad (49)$$

Now substituting  $T(\epsilon)$  and  $g(T)$  into eq.(41) one finds finally

$$f(\epsilon) = \frac{d}{2} - \frac{d}{2} \ln \frac{\pi d}{\epsilon} - \ln 16$$

which is nothing but the leading terms in eq.(32).

Before considering the  $O(\epsilon)$  correction to  $f(\epsilon)$ , let us discuss the form of the surface of singularities  $\tau = \tau_0(\mathbf{x})$ . From eqs.(46), (48) one finds

$$\tau_0(\mathbf{x}) = \frac{\mathbf{x}^2}{T} = \frac{2\epsilon}{d} \mathbf{x}^2 \quad (50)$$

One sees that the surface of singularity has the form of a paraboloid, whose curvature is proportional to the typical momentum of the final particles  $k_0$ . At small  $\epsilon$  the curvature radius is much larger than the inverse boson mass, i.e. 1. One should keep in mind, however, that eq.(50) is valid only for not very large values of  $\mathbf{x}$  (not much larger than  $k^{-1}$ ). At larger  $\mathbf{x}$  the actual behavior of the singularity surface is unknown.

### 3.2.2 Next-to-leading order

To find the first correction to the solution, eq.(43), let us substitute the ansatz

$$\phi_1 = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh(\tau - \tau_0(\mathbf{x}))} + \tilde{\phi}_1(\tau, \mathbf{x})$$

to the field equation. We expect that  $\tilde{\phi}_1$  is suppressed, compared with the leading order, by a factor of  $k_0^2$ , where  $k_0$  is the typical momentum of the final particle, so we drop terms containing  $\phi_1^2$  and  $\phi_1^3$ . We expect also that each derivative with respect to  $\mathbf{x}$  adds an additional factor of  $k_0$ . Denoting  $y = \tau - \tau_0$ , the equation for  $\tilde{\phi}_1$  is linear,

$$\hat{O}\tilde{\phi} \equiv \left[ \partial_\tau^2 + 1 + \frac{6}{\sinh^2 y} \right] \tilde{\phi}_1 = \sqrt{\frac{2}{\lambda}} - \left( \partial_{\mathbf{x}}^2 \tau_0 \cdot \frac{\cosh y}{\sinh^2 y} + (\partial_{\mathbf{x}} \tau_0)^2 \left( \frac{1}{\sinh y} + \frac{2}{\sinh^3 y} \right) \right) \quad (51)$$

We have ignored, for example,  $\partial_{\mathbf{x}} \tilde{\phi}_1$ , since it is of order  $O(k_0^4)$ . The simplest way to solve eq.(51) is to make use of the following simple relations,

$$\begin{aligned} \hat{O} \cosh y &= -6 \frac{\cosh y}{\sinh^2 y}, & \hat{O} \sinh y &= -\frac{6}{\sinh y} \\ \hat{O} \frac{1}{\sinh y} &= -\frac{4}{\sinh^3 y} \end{aligned}$$

to see that one solution to (51) is

$$\sqrt{\frac{2}{\lambda}} \left( \frac{\partial_{\mathbf{x}}^2 \tau_0}{6} \cosh y + \frac{(\partial_{\mathbf{x}} \tau_0)^2}{6} \sinh y + \frac{(\partial_{\mathbf{x}} \tau_0)^2}{2 \sinh y} \right) \quad (52)$$

However, one can also add to (52) an arbitrary solution to the homogeneous equation  $\hat{O}\phi = 0$ . The general solution to the homogeneous equation depends on two arbitrary functions of spatial coordinates,  $F_{1,2}(\mathbf{x})$  and has the form,

$$\phi = F_1(\mathbf{x})f_1(\tau, \mathbf{x}) + F_2(\mathbf{x})f_2(\tau, \mathbf{x})$$

where

$$\begin{aligned} f_1(\tau, \mathbf{x}) &= \frac{\cosh y}{\sinh^2 y} \\ f_2(\tau, \mathbf{x}) &= y \frac{\cosh y}{\sinh^2 y} - \frac{\sinh y}{3} - \frac{1}{\sinh y} \end{aligned} \quad (53)$$

Note that  $f_1$  has a double pole at  $y = \tau - \tau(\mathbf{x}) = 0$ , while  $f_2$  is regular on the singularity surface. As we expect that the field has a pole of the first order at  $y = 0$ , so we should rule out  $f_1$  and set  $F_1(\mathbf{x}) = 0$ <sup>1</sup>. The function  $F_2(\mathbf{x})$  can be chosen from the requirement that  $\phi_1$  decreases in the Euclidean asymptotics  $\tau \rightarrow +\infty$ . In fact, in this limit (52) becomes

$$\sqrt{\frac{2}{\lambda}} \cdot \frac{1}{12} \left( \partial_{\mathbf{x}}^2 \tau_0 + (\partial_{\mathbf{x}} \tau_0)^2 \right) e^{\tau - \tau_0}$$

---

<sup>1</sup>Actually, the term proportional to  $f_1$  arises when one shifts the singularity surface  $\tau_0 \rightarrow \tau_0 + \delta\tau_0$  and try to expand the function  $\sinh^{-1}(\tau - \tau_0 - \delta\tau_0)$  on  $\delta\tau_0$ . Since we want the singularity surface to be  $\tau = \tau_0(\mathbf{x})$ , this term should be excluded.

while the asymptotics of  $f_2$  is also growing

$$f_2(\tau, \mathbf{x}) \sim -\frac{1}{6}e^{\tau-\tau_0}$$

but they cancel each other when we choose  $F_2(\mathbf{x}) = \frac{1}{2}\sqrt{\frac{2}{\lambda}}[\partial_{\mathbf{x}}^2\tau_0 + (\partial_{\mathbf{x}}\tau_0)^2]$ . Therefore, the solution to the field equation with singularities at  $\tau = \tau_0(\mathbf{x})$  and decays at  $\tau \rightarrow +\infty$ , up to the order of  $k_0^2$ , is

$$\begin{aligned} \phi_1 = & \sqrt{\frac{2}{\lambda}} \left[ \frac{1}{\sinh y} + \frac{\partial_{\mathbf{x}}^2\tau_0}{6} \cosh y + \frac{(\partial_{\mathbf{x}}\tau_0)^2}{6} \sinh y + \frac{(\partial_{\mathbf{x}}\tau_0)^2}{2 \sinh y} + \right. \\ & \left. + \frac{\partial_{\mathbf{x}}^2\tau_0 + (\partial_{\mathbf{x}}\tau_0)^2}{2} \left( y \frac{\cosh y}{\sinh^2 y} - \frac{1}{3} \sinh y - \frac{1}{\sinh y} \right) \right] \end{aligned} \quad (54)$$

To find  $a_{\mathbf{k}}$ , in complete analogy with the leading order, one calculates the asymptotics of  $\phi_1$  from eq.(54),

$$\phi_1(\tau, \mathbf{x}) = \sqrt{\frac{2}{\lambda}} \left[ \left( 2 - \frac{5}{6} \partial_{\mathbf{x}}^2\tau_0 \right) e^{-y} + \left( \partial_{\mathbf{x}}^2\tau_0 + (\partial_{\mathbf{x}}\tau_0)^2 \right) y e^{-y} \right] \quad (55)$$

Let us note that, in contrast with the leading order, there is a term proportional to  $ye^{-y}$  in the asymptotics of  $\phi_1$ . This structure also emerges when one evaluate the r.h.s. of eq.(45) to the next-to-leading order, replacing  $e^{\omega_{\mathbf{k}}\tau}$  by  $e^{-\tau}(1 - \frac{\mathbf{k}^2}{2}\tau)$ . Introducing the function

$$a(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{d/2}\sqrt{2\omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}$$

eq.(45) reads

$$\phi_1(\tau, \mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{d/2}\sqrt{2\omega_{\mathbf{k}}}} \left( 1 - \frac{\mathbf{k}^2}{2} \right) a_{\mathbf{k}} e^{-\tau+i\mathbf{k}\mathbf{x}} = \left( a(\mathbf{x}) + \frac{\tau}{2} \partial_{\mathbf{x}}^2 a(\mathbf{x}) \right) e^{-\tau} \quad (56)$$

Comparing eqs.(55) and (56), one finds the relation between  $a(\mathbf{x})$  and  $\tau(x)$ , the next-to-leading-order version of eq.(46),

$$a(\mathbf{x}) = \sqrt{\frac{2}{\lambda}} \left( 2 - \frac{5}{6} \partial_{\mathbf{x}}^2\tau_0 - \tau_0 (\partial_{\mathbf{x}}^2\tau_0 + (\partial_{\mathbf{x}}\tau_0)^2) \right) e^{\tau_0(\mathbf{x})} \quad (57)$$

Let us find out the constraint on  $a(\mathbf{x})$  comes from the requirement that  $\tau_0(0) = 0$ . Taking  $\mathbf{x} = 0$  and noticing that  $\partial_{\mathbf{x}}\tau_0|_{\mathbf{x}=0} = 0$  since the singularity surface is tangent to the plane  $\tau = 0$  at  $\mathbf{x} = 0$ , it is easy to show that this constraint is

$$a(\mathbf{x}) + \frac{5}{12} \partial_{\mathbf{x}}^2 a(\mathbf{x})|_{\mathbf{x}=0} = \sqrt{\frac{8}{\lambda}}$$

In momentum representation this relation reads,

$$\int \frac{d\mathbf{k}}{(2\pi)^{d/2} \sqrt{2\omega_{\mathbf{k}}}} \left(1 - \frac{5}{12} \mathbf{k}^2\right) a_{\mathbf{k}} = \sqrt{\frac{8}{\lambda}} \quad (58)$$

The difference of the constraint from that of the leading order is there is a small correction  $\frac{5}{12} \mathbf{k}^2$ . The calculation is now straightforward and similar to that of the leading order. Introducing the Lagrange multiplier and taking variation with respect to  $a_{\mathbf{k}}$ , one finds

$$a_{\mathbf{k}} = \frac{C}{\sqrt{2\omega_{\mathbf{k}}}} \left(1 - \frac{5}{12} \mathbf{k}^2\right)$$

where  $C$  is a constant that can be found from (58). The final result of the calculations is

$$f(\epsilon) = \frac{d}{2} \left( \ln \frac{\epsilon}{\pi d} + 1 \right) + \ln 16 + \left( \frac{d-2}{4} - \frac{5}{3} \right) \cdot \epsilon$$

which, as expected, coincides with the perturbative result for the tree cross section near threshold. One can, in principle proceed further in this direction and calculate more terms in the expansion of  $f$  on the small parameter  $\epsilon$  by the same technique. In this way one could find  $O(\epsilon^2)$  and higher corrections to eq.(32), which have not been found by standard perturbative techniques. However, let us stop here and turn to the opposite limit  $\epsilon \gg 1$ .

### 3.3 Ultra-relativistic limit in (3+1) dimensions. Lower bound on tree cross section

In the ultra-relativistic limit  $\epsilon \rightarrow \infty$ , the boson mass can be neglected. We will consider the most interesting case  $d = 3$ , where  $\lambda$  is dimensionless. In this case the theory becomes scale invariant and the exponent of the cross section should be independent of the energy (the dependence of the pre-exponent on the energy can be determined solely by dimensional analysis). In other words, we expect that the function  $F(\lambda n, \epsilon)$  becomes independent of  $\epsilon$  at large  $\epsilon$ ,

$$\lim_{\epsilon \rightarrow \infty} F(\lambda n, \epsilon) = F(\lambda n)$$

At tree level, this statement means that  $f(\epsilon)$  has a limit when  $\epsilon \rightarrow \infty$ .

From the first identity in eq.(25) one sees that  $T$  should tends to 0 faster than  $1/\epsilon$  (otherwise  $W = \int dE T(E)$  diverges in the ultraviolet). Now let us take the limit  $\epsilon \rightarrow \infty$  in eq.(41). One finds,

$$f(\infty) = -g(0)$$

Recall that  $\lambda^{-1}e^{g(0)}$  is the minimal value of  $\int d\mathbf{k} a_{\mathbf{k}}^* a_{\mathbf{k}}$  over all solutions of the massless theory that are singular at  $\tau = \mathbf{x} = 0$ . We are unable to find  $g(0)$ . Instead, we will try to bound  $g(0)$  from above. We will take a solution that is known analytically, calculate for it  $a_{\mathbf{k}}$  and plug the result into the definition of  $g(0)$ . If the extremum over  $a_{\mathbf{k}}$  is the true minimum (recall our discussion on the nature of the extremum in subsection 3.1), this procedure would give us an upper bound on  $g(0)$ .

We take the following trial singular solution to the massless field equation [9],

$$\phi(\tau, \mathbf{x}) = \sqrt{\frac{8}{\lambda}} \frac{\rho}{\mathbf{x}^2 + (\tau - \rho)^2 - \rho^2}$$

The Fourier components of this configuration is

$$a_{\mathbf{k}} = \sqrt{\frac{8}{\lambda}} \sqrt{\frac{\pi}{\omega_{\mathbf{k}}}} \rho e^{-\omega_{\mathbf{k}} \rho}$$

which implies the following bound on  $g(0)$ ,

$$g(0) < \ln(8\pi^2)$$

The corresponding lower bound on the cross section is

$$\sigma_n \geq n! \left( \frac{\lambda}{8\pi^2} \right)^n$$

Note that this lower bound grows factorially as  $n \rightarrow \infty$ , which reflects the fact that at tree level not only amplitudes, but also cross sections become large at  $n \sim \lambda^{-1}$ .

## 4 Loop corrections at threshold

### 4.1 General consideration

In this Section we consider another limiting case. Namely, we take arbitrary  $\lambda n$  but small  $\epsilon \ll 1$ , and will be interested in the exponent of the cross section to the leading order of  $\epsilon$ . In this limit, we expect from eq.(7) that the result has the following form,

$$F(\lambda n, \epsilon) = \lambda n \ln \frac{\lambda n}{16} - \lambda n + \frac{d}{2} \left( \ln \frac{\epsilon}{\pi d} + 1 \right) \lambda n + g(\lambda n) \quad (59)$$

where the first term in the expansion of  $g(\lambda n)$  at small  $\lambda n$  should be  $2\text{Re}B\lambda^2 n^2$ . The meaning of the function  $g(\lambda n)$  becomes clear when one recalls that in the limit  $\epsilon \rightarrow 0$

the cross section is the product of the square of the threshold amplitude and the phase volume. Dividing the cross section corresponding to eq.(59) to the phase volume at small  $\epsilon$ , one finds the absolute value of the threshold amplitude,

$$|A_n| \propto n! \left( \frac{\lambda}{8} \right)^{n/2} e^{\frac{1}{\lambda} g(\lambda n)}$$

Comparing with eq.(1), one finds that  $g$  is nothing but the contributions of the loops to the amplitude at threshold. Note that the reproduction of even the leading term in  $g$ ,  $2\text{Re}B\lambda^2 n^2$ , is a nontrivial argument in favor to the validity of the whole semiclassical ideology, since it shows that the exponentiated part of loop contributions is semiclassically calculable.

To find the form of the singularity surface, let us begin by recalling the result of Sect.3 that  $\epsilon \rightarrow 0$  corresponds to  $T \rightarrow \infty$  (eq.(49)), which means that the surface of singularities has a very large curvature radius and in the limit  $\epsilon \rightarrow 0$  can be considered as a plane. However, the discussions in Sect.3 is based on the assumption that the limit  $\lambda n \rightarrow 0$  is taken. When one drops this assumption, one could expect that the presence of the source at  $x = 0$  deforms the surface of singularities near its location. This change should be local and the curvature of the singularity surface should tend to 0 at large  $\mathbf{x}$ . At finite  $\lambda n$ , one expects that the surface of singularities  $\tau = \tau_0(\mathbf{x})$  has the form similar to that sketched in fig.3. The requirement of zero curvature at large distances from  $\mathbf{x} = 0$  can be satisfied if  $\tau_0(\mathbf{x})$  tends to some constant  $\tau_\infty$  as  $\mathbf{x} \rightarrow \infty$ .

If the singularity surface is just the plane  $\tau = \tau_\infty$ , the solution would be  $\mathbf{x}$ -independent and equal to

$$\phi = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh(\tau - \tau_\infty)}$$

In the case when the singularity surface has the form shown in fig.3, the general form of  $\phi$  is

$$\phi = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh(\tau - \tau_\infty)} + \tilde{\phi}(\tau, \mathbf{x}) \quad (60)$$

where  $\tilde{\phi}(\tau, \mathbf{x})$  vanishes at  $\mathbf{x} \rightarrow \infty$  and decays into plane waves in the asymptotics regions, since it comes from the local (about  $\mathbf{x} = 0$ ) deviation of the singularity surface from the plane  $\tau = \tau_\infty$ .

Let us consider the asymptotics of  $\phi$  at  $t \rightarrow +\infty$ . Strictly speaking, the  $\mathbf{x}$ -independent part of  $\phi$  remains non-linear at any  $t$  and does not decays into plane waves. This is an artifact of the limit  $T \rightarrow \infty$ : at any finite  $T$  the field  $\phi$  is a linear superposition of waves at sufficiently large  $t$ . To make  $\phi$  linear, let us take the final part of the contour to form

at a small angle with the Minkowskian time axis. Namely, we take  $t = (1 + i\delta)\alpha$ , where  $\delta$  is small and  $\alpha$  is a real parameter going to infinity. This leads to the replacement

$$\frac{1}{\sinh(\tau - \tau_\infty)} \rightarrow 2e^{(\tau - \tau_\infty)}$$

in the asymptotic region. This part of  $\phi$  is homogeneous on  $\mathbf{x}$ , so its Fourier transformation is a delta-function in the momentum space. In contrast, the Fourier components of  $\tilde{\phi}$  in the asymptotics  $t \rightarrow +\infty$  are supposed to be well behaved functions of  $\mathbf{k}$ . Denoting  $f_{\mathbf{k}}$  and  $g_{\mathbf{k}}$  to be the Feynman and anti-Feynman components of  $\tilde{\phi}$  at the final asymptotics,

$$\tilde{\phi}(t, \mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left( f_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + g_{-\mathbf{k}} e^{i\omega_{\mathbf{k}} t} \right)$$

one finds that the asymptotics of  $\phi$  is

$$\phi(t, \mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[ f_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \left( (2\pi)^{d/2} \sqrt{\frac{16}{\lambda}} \delta(\mathbf{k}) e^{\tau_\infty} + g_{-\mathbf{k}} e^{i\omega_{\mathbf{k}} t} \right) \right] \quad (61)$$

Comparing eqs.(20) and (61), one finds,

$$b_{\mathbf{k}} e^{\omega_{\mathbf{k}} T - \theta} = f_{\mathbf{k}} \quad (62)$$

$$b_{\mathbf{k}}^* = (2\pi)^{d/2} \sqrt{\frac{16}{\lambda}} \delta(\mathbf{k}) e^{\tau_\infty} + g_{\mathbf{k}} \quad (63)$$

From eq.(62) one obtains  $b_{\mathbf{k}} = f_{\mathbf{k}} e^{-\omega_{\mathbf{k}} T + \theta}$ . Since  $T$  is large,  $b_{\mathbf{k}}$  looks like a delta-function in the  $\mathbf{k}$ -space:  $b_{\mathbf{k}}$  is non-zero only in a small region near  $\mathbf{k} = 0$ , namely,  $k \sim T^{-1/2}$ . As  $f_{\mathbf{k}}$  is a smooth function of  $\mathbf{k}$ , one can replace  $f_{\mathbf{k}}$  by its value at  $\mathbf{k} = 0$  and write

$$b_{\mathbf{k}} = f_0 e^{-\omega_{\mathbf{k}} T + \theta}$$

Consider now eq.(63). Since  $b_{\mathbf{k}}^*$  has the form of the delta-function,  $g_{\mathbf{k}}$  vanishes. This important result means that  $\tilde{\phi}(t, \mathbf{x})$  is purely Feynman in the final asymptotics. Eq.(63) now reads

$$f_0^* e^{-\omega_{\mathbf{k}} T + \theta} = (2\pi)^{d/2} \sqrt{\frac{16}{\lambda}} \delta(\mathbf{k}) e^{\tau_\infty} \quad (64)$$

This equation should be understood as a symbolic representation of a relation between  $f_0$ ,  $T$ ,  $\theta$  and  $\tau_\infty$ . To find the latter let us take the integral of both sides of eq.(64) over  $d\mathbf{k}$ . We obtain

$$f_0^* = \sqrt{\frac{16}{\lambda}} T^{d/2} e^{T - \theta + \tau_\infty}$$



The multiplicity and the (small) kinetic energy of the final state can be also expressed in terms of  $T$ ,  $\theta$  and  $\tau_\infty$ ,

$$n = \int d\mathbf{k} b_{\mathbf{k}}^* b_{\mathbf{k}} e^{\omega_{\mathbf{k}} T - \theta} = \frac{16}{\lambda} (2\pi T)^{d/2} e^{T - \theta + 2\tau_\infty} \quad (65)$$

$$n\epsilon = \int d\mathbf{k} \frac{\mathbf{k}^2}{2} b_{\mathbf{k}}^* b_{\mathbf{k}} e^{\omega_{\mathbf{k}} T - \theta} = \frac{16}{\lambda} (2\pi T)^{d/2} e^{T - \theta + 2\tau_\infty} \cdot \frac{d}{2T} \quad (66)$$

Solving eqs.(65) and (66) with respect to  $T$  and  $\theta$ , one finds,

$$T = \frac{d}{2\epsilon}$$

$$\theta = -\ln \frac{\lambda n}{16} + \frac{d}{2} \left( \ln \frac{\pi d}{\epsilon} + 1 \right) + 2\tau_\infty$$

Note that  $T$  and  $\theta$  depend on the surface of singularities through  $\tau_\infty$  but are insensitive of the behavior of the function  $\tau_0(\mathbf{x})$  at finite  $\mathbf{x}$ . In contrast, the action depends on the the precise form of this surface. In Sect.3 we have seen that a part of the action (that is linear on  $\tilde{\phi}$ ) can be reduced to the boundary term  $\tilde{\phi} \partial_t \phi|_{t=\infty}$  and is equal to  $\frac{in}{2}$ . It is convenient to separate this contribution from the action and introduce the notation  $S' = \text{Im}S - \frac{n}{2}$ . The exponent of the cross section is now equal to

$$W = ET - n\theta - 2\text{Im}S = \frac{dn}{2} \left( \ln \frac{\pi d}{\epsilon} + 1 \right) + n \ln \frac{\lambda n}{16} - n - 2S'[\tau_0(\mathbf{x})] - 2n\tau_\infty \quad (67)$$

where  $S'$  a functional depending on  $\tau_0(\mathbf{x})$ . Notice that the first three terms in eq.(67) coincide with those in eq.(59), we conclude that the exponent for the loop corrections to the amplitude at threshold can be obtained by maximizing the expression

$$\frac{1}{\lambda} g(\lambda n) = -n\tau_\infty - S'[\tau_0(\mathbf{x})] \quad (68)$$

with respect to all possible surfaces of singularities  $\tau_0(\mathbf{x})$ . The problem has a simple geometric interpretation: it is equivalent to finding the equilibrium configuration of a surface under the force  $n$  acting to the point  $\mathbf{x} = 0$ , when the energy of the surface depends on its form through the functional  $S'[\tau_0(\mathbf{x})]$ .

Let us summarize our results. To find the amplitude at threshold, one should

1. Find for any function  $\tau = \tau_0(\mathbf{x})$  (which goes to a constant value  $\tau_\infty$  as  $\mathbf{x} \rightarrow \infty$ ) the solution to the field equation which is singular on the surface  $\tau = \tau_0(\mathbf{x})$  and decays at  $\tau \rightarrow +\infty$  and at  $t = +\infty$  has the form (60) where  $\tilde{\phi}$  is purely Feynman.

2. calculate the action of the field, and find  $S' = \text{Im}S - \frac{n}{2}$
3. Extremize the r.h.s. of eq.(68) over all surfaces  $\tau = \tau_0(\mathbf{x})$ .

Unfortunately,  $S'$  seems to be a very complicated functional of  $\tau_0(\mathbf{x})$ . One region when it can be calculated is the regime  $\tau_0 \ll 1$ , where one can develop perturbation theory on  $\tau_0$ . Let us see that this is the regime of low multiplicities,  $n \ll 1/\lambda$ .

## 4.2 Low multiplicities, $\lambda n \ll 1$

When  $\tau_0(\mathbf{x})$  is small the field configuration can be found by solving the field equation separately at  $\tau \ll 1$  and  $\tau \gg \tau_\infty$  and matching in the intermediate region  $\tau_0 \ll \tau \ll 1$ . For convenience we will deform the contour as shown in fig.4, where  $\tau_c$  is some value in the intermediate region,  $\tau_\infty \ll \tau_c \ll 1$ . So, the contour in the  $t$  plane consists of four parts: (I)  $(i\infty, i\tau_c)$ , (II)  $(i\tau_c, 0)$ , (III)  $(0, i\tau_c)$ , (IV)  $(i\tau_c, i\tau_c + \infty)$ . The energy on parts (I) and (II) is 0, while on parts (III) and (IV) of the contour it is equal to  $E$ .

First consider parts (II) and (III). In these part  $\tau \ll 1$ , and the field is given by the following formula,

$$\phi(\tau, \mathbf{x}) = \sqrt{\frac{2}{\lambda}} \frac{1}{\tau - \tau_0(\mathbf{x})} + \text{small corrections} \quad (69)$$

Eq.(69) is valid on both parts (II) and (III). While the leading singular terms coincide, there is difference between  $\phi$  on (II) and (III). Denote

$$\delta\phi(\tau, \mathbf{x}) = \phi_{III}(\tau, \mathbf{x}) - \phi_{II}(\tau, \mathbf{x})$$

Since we expect that this difference is much smaller than the leading term in eq.(69),  $\delta\phi$  satisfies the linearized equation

$$(\partial_\tau^2 + \partial_{\mathbf{x}}^2 + 1 + 3\lambda\phi_0^2)\delta\phi = 0$$

Furthermore, at  $\tau \ll 1$ , the mass and the spatial derivatives can be neglected, so  $\delta\phi$  satisfies the equation  $(\partial_\tau^2 + 3\lambda\phi_0^2)\delta\phi = 0$ , the general solution to which that vanishes on the singularity surface is

$$\delta\phi(\tau, \mathbf{x}) = \sqrt{\frac{2}{\lambda}} W(\mathbf{x})(\tau - \tau_0(\mathbf{x}))^3 \quad (70)$$

where the function  $W(\mathbf{x})$  is yet to be determined.

To the linear order on  $\delta\phi$ , the action on the parts (II) and (III) can be reduced to the boundary term at  $\tau = \tau_c$ ,

$$S'_{II+III} = - \int d\mathbf{x} \partial_\tau \phi \cdot \delta\phi|_{\tau=\tau_c} = \frac{2}{\lambda} \left( (\tau_c - \tau_\infty) \int d\mathbf{x} W(\mathbf{x}) - \int d\mathbf{x} W(\mathbf{x}) c(\mathbf{x}) \right) \quad (71)$$

where  $c(\mathbf{x}) = \tau_0(\mathbf{x}) - \tau_\infty$ . The next correction to  $S'_{II+III}$  is suppressed by a factor of  $\tau_\infty^2$  compared to the leading result (71) and will be neglected.

Consider now the parts (I) and (IV). On these parts of the contours, the solutions to the field equation can be represented in the form of the perturbative series on the background of the homogeneous field,

$$\phi = \phi_{(0)} + \phi_{(1)} + \phi_{(2)} + \dots \quad (72)$$

where  $\phi_{(0)}$  is the  $\mathbf{x}$ -independent solution to the field equation,

$$\phi_0 = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh(\tau - \tau_\infty)} \quad (73)$$

$\phi_{(1)}$  is the linear wave on the background (73) satisfying the equation

$$\partial_\mu^2 \phi_{(1)} - V'(\phi_{(0)}) \phi_{(1)} = 0 \quad (74)$$

everywhere except the singularities and having the following explicit form,

$$\phi_{(1)} = \sqrt{\frac{2}{\lambda}} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{3} c_{\mathbf{k}} f_2^k(\tau) e^{i\mathbf{k}\mathbf{x}}, \quad \text{on part (I)} \quad (75)$$

$$\phi_{(1)} = \sqrt{\frac{2}{\lambda}} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{3} c_{\mathbf{k}} f_1^k(\tau) e^{i\mathbf{k}\mathbf{x}}, \quad \text{on part (IV)} \quad (76)$$

where  $c_{\mathbf{k}}$  are some function of  $\mathbf{k}$ , and  $f_{1,2}^k(\tau)$  are mode functions on the background field (73) [19, 20],

$$f_1^k(\tau) = e^{\omega_{\mathbf{k}}(\tau - \tau_\infty)} \left( \omega_{\mathbf{k}}^2 - \frac{3\omega_{\mathbf{k}}}{\tanh(\tau - \tau_\infty)} + 2 + \frac{3}{\sinh^2(\tau - \tau_\infty)} \right)$$

$$f_2^k(\tau) = f_1^k(2\tau_\infty - \tau)$$

and  $\phi_{(2)}$ , etc. are higher corrections. The small parameter governing the expansion (72) is in fact  $\tau_\infty$ .

The relation between  $c_{\mathbf{k}}$  and the form of the surface of singularities can be found from the matching condition between eqs.(72) and (69) at intermediate values of  $\tau$ . One finds,

$$c_{\mathbf{k}} = \int d\mathbf{x} c(\mathbf{x}) e^{i\mathbf{k}\mathbf{x}}$$

One can also relate  $c_{\mathbf{k}}$  with the function  $W(\mathbf{x})$ . In fact, at  $\tau = \tau_c \ll 1$  one finds from eqs.(75) and (76)

$$\delta\phi_{(1)} = \sqrt{\frac{2}{\lambda}} \frac{(\tau_c - \tau_\infty)^3}{45} \int \frac{d\mathbf{k}}{(2\pi)^d} W_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \quad (77)$$

where

$$W_{\mathbf{k}} = 2\omega_{\mathbf{k}}(\omega_{\mathbf{k}}^2 - 1)(\omega_{\mathbf{k}}^2 - 4). \quad (78)$$

On the other hand, since  $\tau_c \gg \tau_0$  eq.(70) can be expanded on the parameter  $\tau_0/\tau_c$  as follows,

$$\delta\phi = \sqrt{\frac{2}{\lambda}} W(\mathbf{x})(\tau_c - \tau_\infty)^3 - 3W(\mathbf{x})c(\mathbf{x})(\tau_c - \tau_\infty)^2 + \dots \quad (79)$$

The r.h.s. of eq.(77) should coincide with the first term of the r.h.s. of (79), so we obtain the following relation,

$$W(\mathbf{x}) = \frac{1}{45} \int \frac{d\mathbf{k}}{(2\pi)^d} W_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}$$

Let us now evaluate the action on the part (I). Up to the second order of the small parameter the action has the form

$$S'_I = S_I[\phi_0] + \int_{\tau_c}^{\infty} \left( \partial_\mu \phi_{(0)} \partial_\mu \phi_{(1)} + \partial_\mu \phi_{(0)} \partial_\mu \phi_{(2)} + \frac{1}{2} (\partial_\mu \phi_{(1)})^2 + V'(\phi_{(0)})(\phi_{(1)} + \phi_{(2)}) + \frac{1}{2} V''(\phi_{(0)})(\phi_{(1)})^2 \right) \quad (80)$$

Making use of eq.(74) and the fact that  $\phi_{(0)}$  satisfies the field equation, the integral in eq.(80) can be taken in part and the result is

$$S'_I = S_I[\phi_0] - \int d\mathbf{x} \left( (\phi_{(1)} + \phi_{(2)}) \partial_\tau \phi_{(0)} + \frac{1}{2} \phi_{(1)} \partial_\tau \phi_{(1)} \right) \Big|_{\tau=\tau_c}$$

The action on the part (IV) can be calculated in an analogous way. Since  $S_I[\phi_0] + S_{IV}[\phi_0] = 0$ , the sum of the action on parts (I) and (IV) is given by the following boundary terms,

$$S'_{I+IV} = \int d\mathbf{x} \left( \partial_\tau \phi_{(0)} \cdot \delta\phi_{(1)} + \partial_\tau \phi_{(0)} \cdot \delta\phi_{(2)} + \frac{1}{2} \partial_\tau \phi_{(1)} \cdot \delta\phi_{(1)} + \frac{1}{2} \partial_\tau \delta\phi_{(1)} \cdot \phi_{(1)} \right) \Big|_{\tau=\tau_c} \quad (81)$$

The analytical expressions for  $\phi_2$  are rather complicated. Fortunately, the calculation of the action requires only  $\delta\phi_2$  at  $\tau = \tau_c$ , which is equal to the second term in the r.h.s. of eq.(79),

$$\delta\phi_2(\tau_c, \mathbf{x}) = -3W(\mathbf{x})c(\mathbf{x})(\tau_c - \tau_\infty)^2$$

Substitute this to eq.(81) one obtains

$$S'_{I+IV} = \frac{2}{\lambda} \left( -(\tau_c - \tau_\infty) \int d\mathbf{x} W(\mathbf{x}) + \frac{7}{2} \int d\mathbf{x} W(\mathbf{x}) c(\mathbf{x}) \right) \quad (82)$$

The full action can be obtained by taking the sum of (82) and (71). The dependence on  $\tau_c$  disappears, as one can anticipate, and the action is quadratic on  $c_{\mathbf{k}}$ ,

$$S' = \frac{5}{\lambda} \int d\mathbf{x} W(\mathbf{x}) c(\mathbf{x}) = \frac{1}{9\lambda} \int \frac{d\mathbf{k}}{(2\pi)^d} W_{\mathbf{k}} |c_{\mathbf{k}}|^2 \quad (83)$$

Now to extremize (68) we note that

$$\tau_\infty = -c(0) = \int \frac{d\mathbf{k}}{(2\pi)^d} c_{\mathbf{k}} \quad (84)$$

so

$$\frac{1}{\lambda} g = n \int \frac{d\mathbf{k}}{(2\pi)^d} c_{\mathbf{k}} - \frac{1}{9\lambda} \int \frac{d\mathbf{k}}{(2\pi)^d} W_{\mathbf{k}} |c_{\mathbf{k}}|^2 \quad (85)$$

Differentiating  $g$  with respect to  $c_{\mathbf{k}}$ , one obtains

$$c_{\mathbf{k}} = \frac{9\lambda}{2} \frac{n}{W_{\mathbf{k}}} \quad (86)$$

and

$$\frac{1}{\lambda} g = \frac{9\lambda}{4} n^2 \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{W_{\mathbf{k}}} = B\lambda n^2$$

which coincides with the perturbative result for exponentiated leading- $n$  loops.

Let us find the condition for the approximation we use here to be valid. The small parameter is  $\tau_\infty$ , and from eqs.(84) and (86) we see that  $\tau_\infty \sim \lambda n$ . So, our calculations are reliable when  $n$  is small,  $n \ll 1/\lambda$ .

Another remark should be made on the nature of the extremum. Since  $W_{\mathbf{k}}$  does not have definite sign (it is negative at  $\mathbf{k}^2 < 3$  and positive at  $\mathbf{k}^2 > 3$ , see eq.(78)), the extremum of the r.h.s. of (85) is neither maximum nor minimum, but rather a saddle point (in contrast with the maximum for tree-level cross section considered in sect.3). In the theory with broken symmetry  $W_{\mathbf{k}}$  is a positively defined function [21], and we have in this case the true minimum.

## 5 Conclusion

We have seen that the problem of calculation multiparticle cross sections can be reduced to a certain problem of the classical theory. The field configuration describing these

processes is a singular solution to the field equation with certain boundary conditions that optimizes the transition rate. Though most results obtained in this paper can also be found by making use of various perturbative methods, our discussions show that they can be derived from a single approach. We have pointed out an important fact that the exponentiated loop corrections can be calculated semiclassically. We also obtain a new result, namely, the lower bound on tree cross section in the ultra-relativistic regime. Hopefully, the formalism developed in this paper can be used in further analytical or numerical calculations of the multiparticle cross section

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Figure 1: The contour.

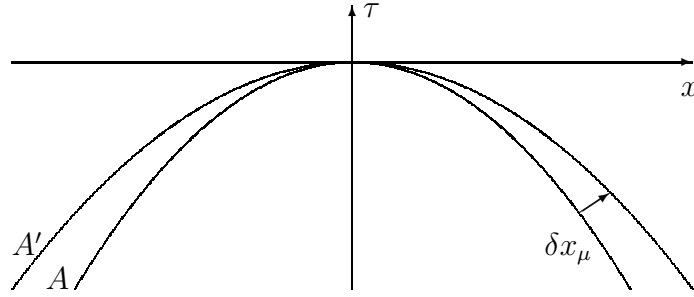


Figure 2: The deformation of the singularity surface.

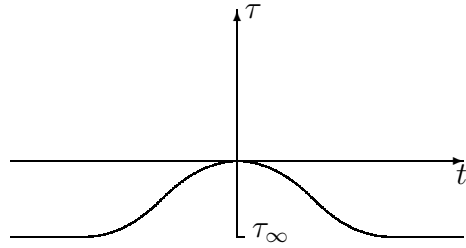


Figure 3: The surface of singularities for configurations describing processes at exact threshold.



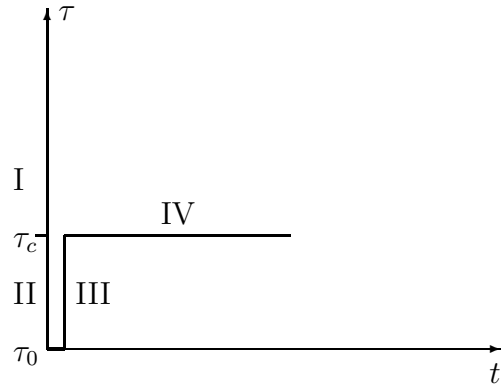


Figure 4: The contour for calculating the action at low multiplicities.